

AN ATRIODIC SIMPLE-4-OD-LIKE CONTINUUM WHICH IS NOT SIMPLE-TRIOD-LIKE

PIOTR MINC

ABSTRACT. The paper contains an example of a continuum K such that K is the inverse limit of simple 4-ods, K cannot be represented as the inverse limit of simple triods and each proper subcontinuum of K is an arc.

1. INTRODUCTION

All topological spaces considered in this paper are metric. A *continuum* is a connected and compact space. A *simple n -od* is the union of n arcs meeting at a common endpoint and which are mutually disjoint otherwise. A simple 3-od is called a *simple triod*. If X and Y are continua, we say that Y is *X -like* provided that Y is the inverse limit of a sequence of copies of X . A continuum is *atriodic* if does not contain three subcontinua A , B and C such that none of them is contained in the union of the remaining two and $\emptyset \neq A \cap B \cap C = A \cap B = A \cap C = B \cap C$.

In 1972, W. T. Ingram gave his brilliant example of an atriodic continuum which is simple-triod-like and not arc-like [4]. Note that an arc-like continuum is simple-2-od-like. The Ingram continuum is not only atriodic, but each of its proper subcontinua is an arc. S. Young asked whether there exists a simple-4-od-like continuum which is not simple-triod-like and whose every proper subcontinuum is an arc [7, Problem 115]. A similar question was asked by H. Cook, W. T. Ingram and A. Lelek. They asked whether there exists an atriodic simple-4-od-like continuum which is not simple-triod-like [1, Problem 5]. Of course, if every proper subcontinuum is an arc, then the continuum is atriodic [3], so a positive answer to Young's question implies a positive answer to the question by Cook, Ingram and Lelek. Even after a perfunctory glance at the problems, it becomes apparent that they should have a positive answer. It is very easy to get an example of a simple-4-od like continuum such that every proper subcontinuum is an arc. Most of such continua appear not to be simple-triod-like and it is very likely that they really are not. So the only difficulty is a proof. Ingram proved that his continuum [4] is not arc-like (chainable) by showing that it has a positive span, and it was proved earlier by Lelek that chainable continua have the span zero [6]. The same method was subsequently

Received by the editors December 27, 1990.

1991 *Mathematics Subject Classification*. Primary 54F15.

Key words and phrases. Continua, simple-triod-like, simple-4-od-like, simplicial maps, graphs, factorization through a triod.

used by Ingram in [5] and by Davis and Ingram in [2]. A topological invariant different than the span is needed to distinguish between those continua which are simple-triod-like and those that are not. Another way of approaching the problem is to use a continuum with simplicial (piecewise linear) bonding maps and prove that they cannot be factored through a simple triod. In this paper we choose this way to prove that there is a simple-4-od-like but not simple-triod-like continuum K such that every proper subcontinuum of K is an arc. Recently, the author [8] introduced an operation d assigning to a simplicial map between graphs a simplicial map between another pair of graphs and using it characterized simplicial maps which can be factored through an arc. This characterization yielded an alternate proof [8, Examples 5.12 and 5.14] of non-chainability for the Ingram and Davis-Ingram continua. In this paper we adapt the same idea to show that some maps cannot be factored through a simple triod.

2. SIMPLICIAL MAPS

In this section we introduce the notion of simplicial maps and prove some auxiliary propositions.

By a *graph* we understand one dimensional, finite simplicial complex. If G is a graph then $\mathcal{V}(G)$ will denote the set of vertices and $\mathcal{E}(G)$ will denote the set of edges. By the order of a vertex v we understand the number of edges containing v . Two points belonging to an edge are called adjacent. A *simplicial map* of a graph G_1 into a graph G_0 is a function from $\mathcal{V}(G_1)$ into $\mathcal{V}(G_0)$ taking every two adjacent vertices either onto a pair of adjacent vertices or onto a single vertex. A simplicial map is *light* if the image of each edge is nondegenerate.

In this paper the same notation is kept for a graph and for its geometric realization. We will assume that every graph is a subset of three dimensional Euclidean space and every edge is a straight linear closed segment between its vertices. In this convention a simplicial map is understood as an actual continuous mapping (linearly extended to the edges). But it is important to note that a graph, either abstract or geometric, has a fixed collection of vertices and any change in this collection changes the graph.

A graph with a geometric realization homeomorphic to an arc is simply called an arc. Observe that two arcs are isomorphic if and only if they have the same number of vertices. A connected graph without a simple closed curve is called a tree. A tree T is a *triod* if it is the union of three arcs intersecting at a common endpoint. If u and v are two adjacent vertices of a graph, by $\langle u, v \rangle$ we will denote the edge between u and v . Additionally, if u and v are two vertices of a tree, by $\langle u, v \rangle$ we will denote the arc between u and v .

2.1 Proposition. *Let $\{G_j, \phi_j^i\}$ be an inverse system of graphs with simplicial and surjective bonding maps $\phi_j^i: G_i \rightarrow G_j$. Let K denote the inverse limit (in the topological sense) of $\{G_j, \phi_j^i\}$. Suppose that K is simple-triod-like, i.e. K is the inverse limit of simple 3-ods with continuous and not necessarily simplicial bonding maps. Then for each positive integer j there is a positive integer i such that ϕ_j^i can be factored through a (simplicial) triod.*

Proof. Let p_n denote the projection of K onto G_n . For each $v \in \mathcal{V}(G_j)$, let $U(v)$ denote a small ball around v in G_j such that $U(v_1) \cap U(v_2) = \emptyset$ for each $v_1, v_2 \in \mathcal{V}(G_j)$ and $v_1 \neq v_2$. Let $\mathcal{U}$ be the open covering of G_j consisting of the sets $U(v)$ and all open edges of G_j . (By an open edge we understand an edge without its endpoints.) The collection $\mathcal{K} = \{p_j^{-1}(U) | U \in \mathcal{U}\}$ is an open covering of K . Since K is simple-triod-like, there is an open covering $\mathcal{F}$ of K such that $\mathcal{F}$ subdivides $\mathcal{K}$ and the nerve of $\mathcal{F}$ is a triod. By a chain of elements of $\mathcal{F}$ we understand a sequence of sets $t_1, t_2, \dots, t_k$ such that $t_n \cap t_m \neq \emptyset$ if and only if $|n - m| \leq 1$. Since the nerve of $\mathcal{F}$ is a triod, for any two elements a and b of $\mathcal{F}$ there is exactly one chain $\text{ch}(a, b)$ with the first element a and the last b . ($\text{ch}(a, a)$ denote the chain reduced to one element a .) If A is a subset of $\mathcal{F}$, then by $\text{conv}(A)$ we will denote the union of all chains $\text{ch}(a, b)$, where $a, b \in A$. Let t be the only element of $\mathcal{F}$ intersecting three other elements of $\mathcal{F}$, and let a_1, a_2 and a_3 be the three elements of $\mathcal{F}$ such that $\text{ch}(t, a_1) \cup \text{ch}(t, a_2) \cup \text{ch}(t, a_3) = \mathcal{F}$.

For each $v \in \mathcal{V}(G_j)$, let $\mathcal{F}(v)$ be the set of the elements of $\mathcal{F}$ contained in $p_j^{-1}(U(v))$. Denote by $\widetilde{\mathcal{F}}$ the union of $\mathcal{F}(v)$, where $v \in \mathcal{V}(G_j)$. We will define an equivalence relation $\cong$ on $\widetilde{\mathcal{F}}$ in the following way: $t_1 \cong t_2$ if there is $v \in \mathcal{V}(G_j)$ such that $t_1, t_2 \in \mathcal{F}(v)$ and $\text{ch}(t_1, t_2) \subset (\mathcal{F} \setminus \widetilde{\mathcal{F}}) \cup \mathcal{F}(v)$. Let Θ denote the set $\widetilde{\mathcal{F}} / \cong$. If $\tau \in \Theta$, then by $v(\tau)$ we will denote the vertex of $\mathcal{V}(G_j)$ such that $\tau \subset \mathcal{F}(v(\tau))$. Observe that if τ_1 and τ_2 are two distinct elements of Θ , then the sets $\text{conv}(\tau_1)$ and $\text{conv}(\tau_2)$ are disjoint. Note also that $\text{conv}(\tau_1 \cup \tau_2) \subset (\mathcal{F} \setminus \widetilde{\mathcal{F}}) \cup \tau_1 \cup \tau_2$ if and only if there are elements $t_1 \in \tau_1$ and $t_2 \in \tau_2$ such that $\text{ch}(t_1, t_2) \subset (\mathcal{F} \setminus \widetilde{\mathcal{F}}) \cup \tau_1 \cup \tau_2$.

Let T be the graph defined in the following way: $\mathcal{V}(T) = \Theta$ and two vertices τ_1 and τ_2 of T are adjacent if $\text{conv}(\tau_1 \cup \tau_2) \subset (\mathcal{F} \setminus \widetilde{\mathcal{F}}) \cup \tau_1 \cup \tau_2$. Let $\beta: T \rightarrow G_j$ be defined by the formula $\beta(\tau) = v(\tau)$ for $\tau \in \mathcal{V}(T)$. Clearly, β is a simplicial map.

We will prove that T is a triod (possibly degenerate). Note that if $\widetilde{\mathcal{F}}$ is contained in the union of two of the chains $\text{ch}(t, a_1)$, $\text{ch}(t, a_2)$ and $\text{ch}(t, a_3)$, then T is an arc (or a single vertex). So we can assume that for each $k = 1, 2, 3$, there is $b_k \in \Theta$ such that b_k intersects the chain $\text{ch}(t, a_k)$ and no other element of Θ intersects $\text{conv}(b_k \cup \{t\})$. Let V_k denote the set of elements of Θ contained in $\text{conv}(b_k \cup \{a_k\})$, and let A_k be the subgraph of T spanned by V_k . Observe that A_k is an arc (possibly degenerate) and b_k is an end point of A_k . Suppose that T is not a triod. Then the vertices b_1, b_2 and b_3 are distinct and each of them is adjacent to the remaining two. Since for each two of the sets $\text{conv}(b_1)$, $\text{conv}(b_2)$ and $\text{conv}(b_3)$ there is a chain in $\mathcal{F}$ between them which does not intersect the third, we may assume that $t \notin \widetilde{\mathcal{F}}$ and $b_k \subset \text{ch}(t, a_k)$ for $k = 1, 2, 3$. Let s_k be the first element of $\text{ch}(t, a_k)$ belonging to $\widetilde{\mathcal{F}}$. Clearly, $s_k \in b_k$. Since $t \notin \widetilde{\mathcal{F}}$, there is an open edge e of G_j such that $p_j(t) \subset e$. Let v' and v'' be the vertices of e . Observe that $p_j(z) \subset e$ for each $z \in \text{ch}(t, s_k) \setminus \{s_k\}$. It follows that $v(s_k)$ is either v' or v'' and consequently two of b_1, b_2 and b_3 coincide. This contradiction proves that T is a triod.

Let ε be a Lebesgue number for the covering $\mathcal{F}$ (i.e. ε is a positive number

such that for each subset Y of K , if the diameter of Y is less than ε , then Y is contained in some element of $\mathcal{T}$. There is a positive integer i such that the diameter $p_i^{-1}(z)$ is less than ε for each $z \in G_i$. Since the bonding maps of the inverse system defining K are surjective, we have that $p_i(p_i^{-1}(z)) = z$. For each $w \in \mathcal{V}(G_i)$, let $a(w)$ be an element of $\mathcal{T}$ containing $p_i^{-1}(w)$. Note that $a(w) \in \mathcal{T}(\varphi_j^i(w))$. Let $\alpha(w)$ be the vertex of T representing $a(w)$. Clearly, $\beta \circ \alpha = \varphi_j^i$. To complete the proof it is enough to show that α is a simplicial map.

Let w and w' be two adjacent vertices of G_i . We will prove that $\alpha(w)$ and $\alpha(w')$ are adjacent vertices of T . Let $I_1, I_2, \dots, I_n$ be a chain covering of $\langle w, w' \rangle$ such that $p_i^{-1}(I_1) \subset a(w)$, $p_i^{-1}(I_n) \subset a(w')$ and the diameter $p_i^{-1}(I_k)$ is less than ε for each $k = 1, \dots, n$. Let B_k be an element of $\mathcal{T}$ containing $p_i^{-1}(I_k)$ with $B_1 = a(w)$ and $B_n = a(w')$.

Consider two cases $\varphi_j^i(w) = \varphi_j^i(w')$ and $\varphi_j^i(w) \neq \varphi_j^i(w')$. If $\varphi_j^i(w) = \varphi_j^i(w')$, then $\varphi_j^i(I_k) = \varphi_j^i(w)$ and thus $B_k \in \mathcal{T}(\varphi_j^i(w))$ for each $k = 1, \dots, n$. It follows that in this case $\alpha(w) = \alpha(w')$. So we may assume that $\varphi_j^i(w) \neq \varphi_j^i(w')$. Let e be the edge between $\varphi_j^i(w)$ and $\varphi_j^i(w')$. Since $p_j(B_k) \cap e \neq \emptyset$, we have the result that $p_j(B_k) \subset e \cup U(\varphi_j^i(w)) \cup U(\varphi_j^i(w'))$ and thus $B_k \in (\mathcal{T} \setminus \widetilde{\mathcal{T}}) \cup \mathcal{T}(\varphi_j^i(w)) \cup \mathcal{T}(\varphi_j^i(w'))$. Let m be the greatest integer such that $B_m \in \mathcal{T}(\varphi_j^i(w))$. Since $\varphi_j^i(I_1 \cup \dots \cup I_m) \subset U(\varphi_j^i(w))$ and $U(\varphi_j^i(w)) \cap U(\varphi_j^i(w')) = \emptyset$, we have the result that $B_k \in (\mathcal{T} \setminus \widetilde{\mathcal{T}}) \cup \mathcal{T}(\varphi_j^i(w))$ for each $k = 1, \dots, m$. It follows that $B_1 \cong B_m$. Let m' be the least integer such that $B_{m'} \in \mathcal{T}(\varphi_j^i(w'))$. Clearly, $m' > m$. By the same argument as the one above we infer that $B_{m'} \cong B_n$. Since the collection $B_m, B_{m+1}, \dots, B_{m'}$ contains the chain $\text{ch}(B_m, B_{m'})$, we have the result that $\alpha(w)$ and $\alpha(w')$ are adjacent vertices of T . $\square$

We need to recall the following definitions from [8, 5.1 and 5.3].

2.2 Definition. We will say that a graph G' *subdivides* a graph G if G' is a graph obtained from G by adding vertices on some of its edges. More precisely, G' is a graph such that $\mathcal{V}(G) \subset \mathcal{V}(G')$ and for every edge $e \in \mathcal{E}(G)$ there is an arc (e, G') contained in G' such that

- (i) (e, G') has the same endpoints as e ,
- (ii) $(d, G') \cap (e, G') = d \cap e$ for $d, e \in \mathcal{E}(G)$ and $d \neq e$, and
- (iii) every vertex from $\mathcal{V}(G')$ belongs to some (e, G') and every edge from $\mathcal{E}(G')$ is an edge of some (e, G') .

If v is a vertex of G and e is an edge of G containing v , then by (v, e, G') we denote the edge of (e, G') containing v .

Let $\varphi: G_1 \rightarrow G_0$ be a simplicial map between graphs. Let G'_0 be a graph subdividing G_0 and let φ' be a simplicial map of a graph G'_1 subdividing G_1 onto G'_0 . We will say that φ' is a *subdivision* of φ *matching* G'_0 provided that $\varphi'(v) = \varphi(v)$ for each vertex $v \in \mathcal{V}(G_1)$, and for each edge $e \in \mathcal{E}(G_1)$ we have that

- if $\varphi(e)$ is degenerate then $(e, G'_1) = e$, and
- if $\varphi(e)$ is an edge of G_0 then φ' is an isomorphism of (e, G'_1) onto $(\varphi(e), G_0)$.

2.3 Proposition. *Suppose $\psi: G_1 \rightarrow G_0$ is a simplicial map between connected graphs. Let G'_0 be a graph subdividing G_0 and let $\psi': G'_1 \rightarrow G'_0$ be a subdivision of ψ matching G'_0 . Then ψ can be factored through a triod if and only if ψ' can be factored through a triod.*

Proof. Observe that by [8, Proposition 5.4], if ψ can be factored through a triod, then ψ' also can be factored through a triod. Suppose that there is a triod T' and there are simplicial maps $\alpha': G'_1 \rightarrow T'$ and $\beta': T' \rightarrow G'_0$ such that $\beta' \circ \alpha' = \psi'$. In view of [8, Proposition 5.13] we can assume that α' is surjective. Let t denote the only order 3 vertex of T' . Let t_0, t_1 and t_2 be the endpoints of T' . Let $V = \{v \in \mathcal{V}(T') \mid \beta'(v) \in \mathcal{V}(G_0)\}$. Observe that $\alpha'(\mathcal{V}(G'_1)) = V$ and $\alpha'(\mathcal{V}(G'_1) \setminus \mathcal{V}(G_1)) = \mathcal{V}(T') \setminus V$. Let $V_i = V \cap \langle t, t_i \rangle$ and let w_i be the vertex of V_i which is the closest to t . Let A_i denote the graph with V_i as its set of vertices such that any two vertices of V_i are adjacent if there are no other points of V between them. Observe that A_i is an arc. We will prove the following claim.

Claim. We can assume that $\beta'(w_0) = \beta'(w_2)$.

If $t \in V$, then $t = w_0 = w_1 = w_2$ and the claim is true. So we can assume that $t \notin V$. Since α' is surjective and G_1 is connected there are two pairs a, a' and b, b' of vertices of G_1 such that a and a' are adjacent in G_1 , b and b' are adjacent in G_1 , and the set $\{\alpha'(a), \alpha'(a'), \alpha'(b), \alpha'(b')\}$ consists of all three vertices w'_0, w_1 and w_2 .

Without loss of generality we can assume that $\alpha'(a) = w_0$, $\alpha'(a') = w_1 = \alpha'(b)$ and $\alpha'(b') = w_2$. Let e_0 be the edge of G_0 joining $\beta'(w_0) = \beta'(\alpha'(a)) = \psi(a)$ and $\beta'(w_1) = \beta'(\alpha'(a')) = \psi(a')$. Let e_1 be the edge of G_0 joining $\beta'(w_1) = \beta'(\alpha'(b)) = \psi(b)$ and $\beta'(w_2) = \beta'(\alpha'(b')) = \psi(b')$. Since (e_0, G'_0) and (e_1, G'_0) have two common vertices $\beta'(w_1)$ and $\beta'(t)$, e_0 and e_1 coincide. Thus $\beta'(w_0) = \beta'(w_2)$.

We will define T considering two cases $\beta'(w_0) = \beta'(w_2) = \beta'(w_1)$ and $\beta'(w_0) = \beta'(w_2) \neq \beta'(w_1)$. In the first case T is the union of A_0, A_1 and A_2 with w_0, w_1 and w_2 identified to one vertex. If $\beta'(w_0) = \beta'(w_2) \neq \beta'(w_1)$, T is the union of A_0, A_1 and A_2 with w_0 and w_2 identified to one vertex, and with an edge between the result of the identification and w_1 .

Let $\beta: T \rightarrow G_0$ be such that $\beta(v) = \beta'(v)$ for each $v \in V$. Note that β is a simplicial map. Let $\alpha: G_1 \rightarrow T$ be such that $\alpha(v) = \alpha'(v)$ for each $v \in \mathcal{V}(G_1)$. One can verify that α is a simplicial map and $\beta \circ \alpha = \psi$. $\square$

3. CONSTRUCTION OF THE EXAMPLE

In this section we will construct a simplicial map φ between two subdivisions of a simple 4-od. Then we consider the inverse system with subdivisions of φ as bonding maps. We define K to be the inverse limit of this system. We show in this section that each proper subcontinuum of K is an arc. We will show later that K is not simple-triod-like.

Let X be a tree with its vertices named as in Figure 1.

Let X' be a subdivision of X with twelve new vertices $u_0, u_1, \dots, u_{11}$ added as shown in Figure 2. Let $\varphi: X' \rightarrow X$ be the simplicial map defined by Table 1 and the following equality $\varphi(u_0) = \varphi(u_2) = \varphi(u_4) = \varphi(u_5) = \varphi(u_7) = \varphi(u_9) = \varphi(u_{10}) = v_0$.

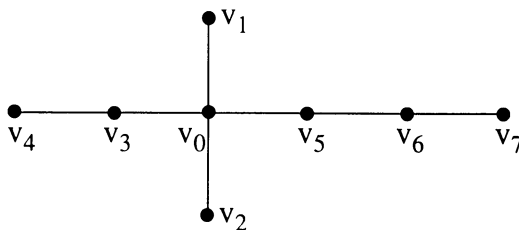


FIGURE 1

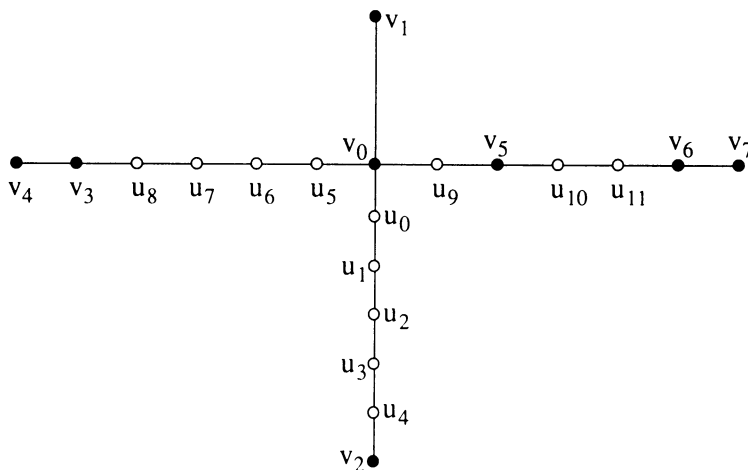


FIGURE 2

TABLE 1

$x :$	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	u_1	u_3	u_6	u_8	u_{11}
$\varphi(x) :$	v_3	v_4	v_1	v_6	v_7	v_1	v_6	v_7	v_5	v_2	v_2	v_5	v_5

The map φ is indicated in Figure 3. The dotted line graph represents the domain, X' , while the solid black represents the range, X , and each vertex of the domain is mapped onto the nearest vertex of the range. Only vertices $v_0, v_1, \dots, v_7$ of X' are labeled. Not labeled are the remaining vertices of X' and the vertices of X , which are named exactly the same as shown in Figure 1.

3.1 Proposition. $\varphi^4(\langle u_{11}, v_6 \rangle) = X$ and for each vertex x of X' , $\varphi^4(x)$ is either v_6 or v_7 . (φ^n is the n th iteration of φ .) $\square$

We need to recall another definition from [8].

3.2 Definition. Let n be a positive integer and let N denote either the set $\{0, 1, \dots, n\}$ or the set of all nonnegative integers. Denote by N_1 the set $N \setminus \{0\}$. Let $G_0, G_1, G_2, \dots$ be a sequence of graphs with N as the set of indices. Let Σ be a sequence of simplicial maps $\psi_1, \psi_2, \dots$ such that for each $j \in N_1$, ψ_j maps a graph G'_j subdividing G_j into G_{j-1} . Using inductively Proposition 5.4 from [8], we can define a sequence of simplicial maps $\sigma_1, \sigma_2, \dots$ such that $\sigma_1 = \psi_1$ and for each $j \in N_1 \setminus \{1\}$, σ_j subdivides ψ_j

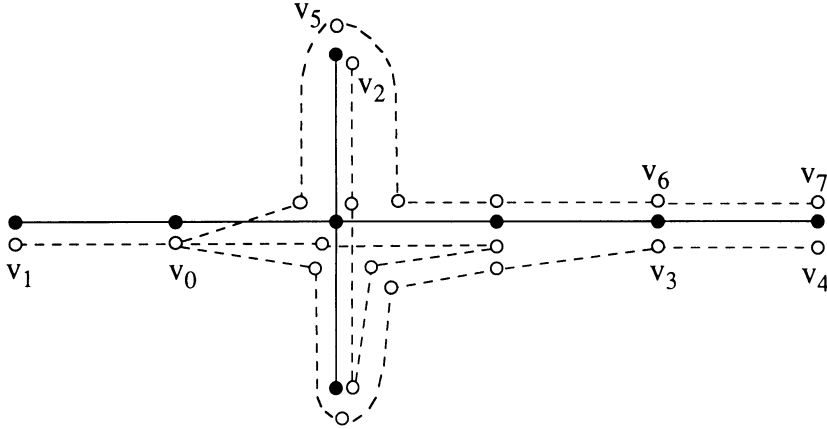


FIGURE 3

matching the domain of σ_{j-1} . For each $j \in N_1$, denote by Σ_j the domain of ψ_j . Set $\Sigma_0 = G_0$. For every two integers i and j from N such that $i > j$, let Σ_j^i denote the composition $\sigma_{j+1} \circ \cdots \circ \sigma_i$ mapping Σ_i into Σ_j . By Σ_j^i we will understand the identity on Σ_i . We will say that the inverse system $\{\Sigma_j, \Sigma_j^i\}$ is generated by the sequence Σ .

Let Φ be an infinite sequence of simplicial maps $\varphi_1, \varphi_2, \dots$ each of which is φ . By $\{\Phi_j, \Phi_j^i\}$ we denote the inverse system generated by Φ . Let K be the inverse limit of $\{\Phi_j, \Phi_j^i\}$. Denote by p_j the projection of K onto Φ_j .

3.3 Proposition. *Every proper nondegenerate subcontinuum of K is an arc.*

Proof. Let P be a proper subcontinuum of K . Suppose that P is not an arc. Since φ is simplicial on X' , there is a vertex x of X' such that $x \in p_j(P)$ for infinitely many j . In view of Proposition 3.1, we have that either $v_6 \in p_j(P)$ for all j or $v_7 \in p_j(P)$ for all j . Since φ restricted to $\langle u_{11}, v_7 \rangle$ is an embedding, $u_{11} \in p_j(P)$ for infinitely many j . Since $\langle u_{11}, v_6 \rangle \subset \langle u_{11}, v_7 \rangle$ and $\varphi^4(\langle u_{11}, v_6 \rangle) = X$, $p_j(P) = X$ for each j . Thus P is not proper. $\square$

3.4 Proposition. *Let n be a positive integer. Let e be an edge of X . Let $\tilde{e}$ denote the interior of e . Suppose $C \subset \Phi_n$ is a component of $(\Phi_1^n)^{-1}(\tilde{e})$. Then the closure of C is mapped by Φ_1^n isomorphically onto e .*

Proof. Since Φ_0^{n-1} is a simplicial map onto X , any component of $(\Phi_0^{n-1})^{-1}(\tilde{e})$ is the interior of an edge of Φ_{n-1} . Since Φ_1^n is a simplicial subdivision of Φ_0^{n-1} matching X' , C is a subdivision of the interior of an edge of Φ_{n-1} . Since Φ_0^{n-1} maps isomorphically edges of Φ_{n-1} onto edges of X , C is mapped by Φ_1^n isomorphically onto e . $\square$

3.5 Proposition. *Let n be a positive integer. Suppose that a is a vertex of Φ_n such that $\Phi_0^n(a) = v_0$. Then a is a vertex of order 2 in Φ_n . Moreover, if b and c are the two vertices of Φ_n adjacent to a , then Φ_0^n is an embedding of $\langle b, c \rangle$ into X .*

Proof. Since $\Phi_1^n(a)$ is a vertex of X' but not of X , the proposition follows from Proposition 3.4.

3.6 Proposition. Suppose that Y is a triod which is the union of three arcs A_1 , A_2 and A_3 meeting at a common endpoint y . Let $y, y_1^1, y_2^1, \dots, y_{k(i)}^1$ denote the sequence of consecutive vertices of A_i . Let $\beta: Y \rightarrow X$ be a simplicial map such that $\beta(y) = v_0$ and the points $\beta(y_1^1)$, $\beta(y_2^1)$ and $\beta(y_3^1)$ are three different vertices from the set $\{v_1, v_2, v_3, v_5\}$. Suppose there is a positive integer n and there is a simplicial map $\alpha: \Phi_n \rightarrow Y$ such that $\beta \circ \alpha = \Phi_0^n$. Then there is a triod Y' with its vertex of order three denoted by y' and there are simplicial maps $\beta': Y' \rightarrow X$ and $\alpha': \Phi_n \rightarrow Y'$ such that $\beta' \circ \alpha' = \Phi_0^n$ and $\beta'(y') \neq v_0$.

Proof. Let $W_{1,2}$, $W_{1,3}$ and $W_{2,3}$ denote the sets $\alpha^{-1}(A_1) \cap \alpha^{-1}(A_2)$, $\alpha^{-1}(A_1) \cap \alpha^{-1}(A_3)$ and $\alpha^{-1}(A_2) \cap \alpha^{-1}(A_3)$, respectively. By Proposition 3.5, the sets $W_{1,2}$, $W_{1,3}$ and $W_{2,3}$ are mutually exclusive and $\alpha^{-1}(y)$ is their union.

3.6.1 Claim. If one of the sets $W_{1,2}$, $W_{1,3}$ and $W_{2,3}$ is empty then the proposition is true.

Since A_1 , A_2 and A_3 play the same role in the statement of the proposition, we may assume that $W_{1,2} = \emptyset$. Define $\mathcal{Z}(Y')$ to be $\mathcal{Z}(Y)$ with y replaced by two points w_1 and w_2 . Let $\mathcal{Z}(Y')$ consists of $\langle y_1^3, w_1 \rangle$, $\langle y_1^3, w_2 \rangle$, $\langle w_1, y_1^1 \rangle$, $\langle w_2, y_1^2 \rangle$ and all edges of Y not containing y . Define $\beta'(w_1) = \beta'(w_2) = v_0$ and $\beta'(w) = \beta(w) = \alpha(v)$ for $w \in \mathcal{Z}(Y') \cap \mathcal{Z}(Y)$. Define $\alpha'(v)$ for $v \in \mathcal{Z}(\Phi_n) \setminus \alpha^{-1}(y)$, $\alpha'(v) = w_1$ for $v \in W_{1,3}$ and $\alpha'(v) = w_2$ for $v \in W_{2,3}$. One can verify that the hypothesis of the proposition is satisfied with $y' = y_1^3$.

3.6.2 Claim. Suppose that $\beta(y_1^1) = v_1$ and $\beta(y_2^1) = v_2$. Suppose also that either $\beta(y_3^1) = v_3$ or $\beta(y_3^1) = v_5$ and $\beta(y_2^3) = v_6$. If $W_{1,2} \neq \emptyset$, then $\alpha(\Phi_n) \subset A_1 \cup A_2$.

By Proposition 3.5, there are three vertices w_1 , w and w_2 of Φ_n such that w_1 is adjacent to w , w is adjacent to w_2 , $\alpha(w_1) = y_1^1$, $\alpha(w) = y$ and $\alpha(w_2) = y_2^2$. Since $v_2 - u_4 - u_3$ is the only pair of intersecting edges of X' mapped by φ onto $v_1 - v_0 - v_2$, we have the result that $\Phi_1^n(w_1) = v_2$, $\Phi_1^n(w) = u_4$ and $\Phi_1^n(w_2) = u_3$. Let x be an arbitrary vertex of Φ_n . Let $s_0 = w$, $s_1, \dots, s_k = x$ be vertices of $\langle w, x \rangle$ such that for $i = 0, 1, \dots, k-1$, $\alpha(s_i) = y$ and $\alpha(a) \neq y$ for any vertex a from the interior of $\langle s_i, s_{i+1} \rangle$.

In order to prove the claim, it is enough to show that if $\Phi_1^n(s_i) = u_4$, then $\alpha(\langle s_i, s_{i+1} \rangle) \subset A_1 \cup A_2$ and if $i+1 < k$, then also $\Phi_1^n(s_{i+1}) = u_4$. Observe that $\alpha(\langle s_i, s_{i+1} \rangle)$ is contained in one of the arcs A_1 , A_2 and A_3 . Let $a_0 = s_i$, $a_1, a_2, \dots, a_m = s_{i+1}$ be vertices of $\langle s_i, s_{i+1} \rangle$ listed in the natural order. Clearly, either $\Phi_1^n(a_1) = u_3$ or $\Phi_1^n(a_1) = v_2$. We will consider each of these cases separately.

Suppose $\Phi_1^n(a_1) = u_3$. Since $\beta \circ \alpha = \Phi_0^n = \varphi \circ \Phi_1^n$, $\alpha(a_1) = y_1^2$ and consequently $\alpha(\langle s_i, s_{i+1} \rangle) \subset A_2$. By Proposition 3.4, there is a vertex b of Φ_n such that Φ_0^n maps consecutive vertices of $\langle a_1, b \rangle$ onto the sequence v_2, v_0, v_5, v_0, v_3 . To match this pattern, α must map $\langle a_1, b \rangle$ into A_2 and we must have that $\beta(y_2^2) = v_0$, $\beta(y_3^2) = v_5$, $\beta(y_4^2) = v_0$ and $\beta(y_5^2) = v_3$. Now, assume that $\alpha(s_{i+1}) = y$. Clearly, $m > 5$ and $\alpha(a_2) = y_2^2$, $\alpha(a_3) = y_3^2$, $\alpha(a_4) = y_4^2$ and $\alpha(a_5) = y_5^2$. Let $j < m$ be the greatest integer such that $\alpha(a_j) = y_4^2$. Note that $m \geq j+4$. Since $\beta \circ \alpha = \Phi_0^n$, it follows from Proposition 3.5 that $\alpha(a_{j-1}) = y_5^2$ and $\alpha(a_{j+1}) = y_3^2$. Observe that $\Phi_0^n(a_{j-1}) = v_3$, $\Phi_0^n(a_j) = v_0$ and $\Phi_0^n(a_{j+1}) = v_5$. Since $v_0 - u_0 - u_1$ is the only pair of intersecting edges of X' mapped by φ onto $v_3 - v_0 - v_5$, $\Phi_1^n(a_{j+1}) = u_1$. By Propo-

sition 3.4, there is a vertex c of Φ_n such that $a_j \in \langle a_{j-1}, c \rangle$ and $\langle a_{j-1}, c \rangle$ is mapped by Φ_1^n isomorphically onto $\langle v_0, v_2 \rangle$. It follows from Proposition 3.4 that the vertices a_{j+1} , a_{j+2} , a_{j+3} and a_{j+4} belong to $\langle a_{j-1}, c \rangle$. Since $\langle a_{j-1}, c \rangle$ has seven vertices, a_{j+4} is adjacent to c . Since $\alpha(a_{j-1}) = y_5^2$, $\alpha(a_j) = y_4^2$, $\alpha(a_{j+1}) = y_3^2$ and Φ_0^n maps consecutive vertices of $\langle a_{j-1}, c \rangle$ onto the sequence $v_3, v_0, v_5, v_0, v_2, v_0, v_1$, we have the result that $\alpha(c) = y_1^1$. Thus $\alpha(a_{j+4}) = y$, $s_{i+1} = a_{j+4}$ and consequently $\Phi_1^n(s_{i+1}) = u_4$. This completes the proof of the claim in the case where $\Phi_1^n(a_1) = u_3$.

Suppose $\Phi_1^n(a_1) = v_2$. The proof in this case is essentially the same as in the previous case. Since $\beta \circ \alpha = \Phi_0^n = \varphi \circ \Phi_1^n$, $\alpha(a_1) = y_1^1$ and consequently $\alpha(\langle s_i, s_{i+1} \rangle) \subset A_1$. Now, suppose that $\alpha(s_{i+1}) = y$. Clearly, $m \geq 2$ and $\Phi_1^n(a_2) = u_4$. Observe that either $\alpha(a_2) = y$ or $\alpha(a_2) = y_2^1$. In the first case the claim is satisfied, so we can assume that $\alpha(a_2) = y_2^1$. By Proposition 3.4, there is a vertex b of Φ_n such that Φ_0^n maps consecutive vertices of $\langle a_1, b \rangle$ onto the sequence $v_1, v_0, v_2, v_0, v_5, v_0, v_3$. To match this pattern, α must map $\langle a_1, b \rangle$ into A_1 and must have that $\beta(y_2^1) = v_0$, $\beta(y_3^1) = v_2$, $\beta(y_4^1) = v_0$, $\beta(y_5^1) = v_5$, $\beta(y_6^1) = v_0$ and $\beta(y_7^1) = v_3$. Clearly, $m > 7$ and $\alpha(a_3) = y_3^1$, $\alpha(a_4) = y_4^1$, $\alpha(a_5) = y_5^1$, $\alpha(a_6) = y_6^1$ and $\alpha(a_7) = y_7^1$. Let $j < m$ be the greatest integer such that $\alpha(a_j) = y_6^1$. Note that $m \geq j+6$. Since $\beta \circ \alpha = \Phi_0^n$, it follows from Proposition 3.5 that $\alpha(a_{j-1}) = y_7^1$ and $\alpha(a_{j+1}) = y_5^1$. Observe that $\Phi_0^n(a_{j-1}) = v_3$, $\Phi_0^n(a_j) = v_0$ and $\Phi_0^n(a_{j+1}) = v_5$. Since $v_0 - u_0 - u_1$ is the only pair of intersecting edges of X' mapped by φ onto $v_3 - v_0 - v_5$, $\Phi_1^n(a_{j+1}) = u_1$. By Proposition 3.4, $\langle a_{j-1}, a_{j+5} \rangle$ is mapped by Φ_1^n isomorphically onto $\langle v_0, v_2 \rangle$. In particular $\Phi_1^n(a_{j+5}) = v_2$ and consequently $\Phi_1^n(a_{j+6}) = u_4$. Since $\alpha(a_{j-1}) = y_7^1$, $\alpha(a_j) = y_6^1$, $\alpha(a_{j+1}) = y_5^1$ and Φ_0^n maps consecutive vertices of $\langle a_{j-1}, a_{j+5} \rangle$ onto the sequence $v_3, v_0, v_5, v_0, v_2, v_0, v_1$, we have the result that $\alpha(a_{j+5}) = y_1^1$. If $\alpha(a_{j+6}) = y$, then $s_{i+1} = a_{j+6}$ and the claim is true. Suppose that $\alpha(a_{j+6}) \neq y$ and thus $\alpha(a_{j+6}) = y_2^2$. By Proposition 3.4, there is a vertex c of Φ_n such that $a_{j+6} \in \langle a_{j+5}, c \rangle$ and $\langle a_{j+5}, c \rangle$ is mapped by Φ_1^n isomorphically onto $\langle v_0, v_2 \rangle$. Since $\alpha(a_{j+5}) = y_1^2$, $\alpha(a_{j+6}) = y_2^1$ and Φ_0^n maps consecutive vertices of $\langle a_{j+5}, c \rangle$ onto the sequence $v_1, v_0, v_2, v_0, v_5, v_0, v_3$, we have the result that $\alpha(\langle a_{j+5}, c \rangle) = \langle y_1^1, y_7^1 \rangle$. It follows that $m > j+11$ and $\alpha(a_{j+10}) = y_6^1$. This contradicts the choice of j . So the claim is true.

We will consider the following four cases: Case (i). $\beta(y_1^1) = v_1$, $\beta(y_1^2) = v_2$ and $\beta(y_1^3) = v_3$, Case (ii). $\beta(y_1^1) = v_1$, $\beta(y_1^2) = v_2$ and $\beta(y_1^3) = v_5$, Case (iii). $\beta(y_1^1) = v_1$, $\beta(y_1^2) = v_3$ and $\beta(y_1^3) = v_5$, and Case (iv). $\beta(y_1^1) = v_2$, $\beta(y_1^2) = v_3$ and $\beta(y_1^3) = v_5$.

Case (i). $\beta(y_1^1) = v_1$, $\beta(y_1^2) = v_2$ and $\beta(y_1^3) = v_3$.

By 3.6.2, if $W_{1,2} \neq \emptyset$, then $\alpha(\Phi_n)$ is an arc and the proposition is trivially satisfied. So, we can assume that $W_{1,2} = \emptyset$ and infer the proposition from 3.6.1.

Case (ii). $\beta(y_1^1) = v_1$, $\beta(y_1^2) = v_2$ and $\beta(y_1^3) = v_5$.

If $W_{1,3} = \emptyset$, then the proposition is true by 3.5.1. So we can assume that $W_{1,3} \neq \emptyset$. By Proposition 3.5, there are three vertices w_1 , w and w_3 of Φ_n such that w_1 is adjacent to w , w is adjacent to w_3 , $\alpha(w_1) = y_1^1$, $\alpha(w) = y$ and $\alpha(w_3) = y_1^3$. Since $v_5 - u_{10} - u_{11}$ is the only pair of intersecting edges of X' mapped by φ onto $v_1 - v_0 - v_5$, we have the result that $\Phi_1^n(w_1) = v_5$, $\Phi_1^n(w) = u_{10}$ and $\Phi_1^n(w_3) = u_{11}$. By Proposition 3.4, there is a vertex c of Φ_n

such that c is adjacent to w_3 and $\Phi_1^n(c) = v_6$. Since $\Phi_0^n(c) = v_6$, $\alpha(c) = y_2^3$ and $\beta(y_2^3) = v_6$. Now, we can use 3.6.2. If $W_{1,2} \neq \emptyset$, then $\alpha(\Phi_n)$ is an arc and the proposition is trivially satisfied. So, we can assume that $W_{1,2} = \emptyset$ and infer the proposition from 3.6.1.

Case (iii). $\beta(y_1^1) = v_1$, $\beta(y_1^2) = v_3$ and $\beta(y_1^3) = v_5$.

If $W_{1,3} = \emptyset$, then the proposition is true by 3.6.1. So we can assume that $W_{1,3} \neq \emptyset$. By Proposition 3.5, there are three vertices w_1 , w and w_3 of Φ_n such that w_1 is adjacent to w , w is adjacent to w_3 , $\alpha(w_1) = y_1^1$, $\alpha(w) = y$ and $\alpha(w_3) = y_1^3$. Since $v_5 - u_{10} - u_{11}$ is the only pair of intersecting edges of X' mapped by φ onto $v_1 - v_0 - v_5$, we have the result that $\Phi_1^n(w_1) = v_5$, $\Phi_1^n(w) = u_{10}$ and $\Phi_1^n(w_3) = u_{11}$. By Proposition 3.4, there is a vertex c of Φ_n such that c is adjacent to w_3 and $\Phi_1^n(c) = v_6$. Since $\Phi_0^n(c) = v_6$, $\alpha(c) = y_2^3$ and $\beta(y_2^3) = v_6$.

If $W_{2,3} = \emptyset$, then the proposition is true by 3.6.1. So we can assume that $W_{2,3} \neq \emptyset$. By Proposition 3.5, there are three vertices u_2 , u and u_3 of Φ_n such that u_2 is adjacent to u , u is adjacent to u_3 , $\alpha(u_2) = y_1^2$, $\alpha(u) = y$ and $\alpha(u_3) = y_1^3$. Since $v_0 - u_0 - u_1$ is the only pair of intersecting edges of X' mapped by φ onto $v_3 - v_0 - v_5$, we have the result that $\Phi_1^n(u_2) = v_0$, $\Phi_1^n(u) = u_0$ and $\Phi_1^n(u_3) = u_1$. By Proposition 3.4, there are vertices a and b of Φ_n such that a is adjacent to u_3 , b is adjacent to a , $\Phi_1^n(a) = u_2$ and $\Phi_1^n(b) = u_3$. Since $\Phi_0^n(a) = v_0$ and $\beta(y_2^3) = v_6$, we have the result that $\alpha(a) = y$. Thus $\alpha(b)$ must be one of the points y_1^1 , y_1^2 and y_1^3 . This is a contradiction, because $\beta(\alpha(b)) = \Phi_0^n(b) = v_2$, $\beta(y_1^1) = v_1$, $\beta(y_1^2) = v_3$ and $\beta(y_1^3) = v_5$.

Case (iv). $\beta(y_1^1) = v_2$, $\beta(y_1^2) = v_3$ and $\beta(y_1^3) = v_5$.

If $W_{2,3} = \emptyset$, then the proposition is true by 3.6.1. So we can assume that $W_{2,3} \neq \emptyset$. By Proposition 3.5, there are three vertices w_2 , w and w_3 of Φ_n such that w_2 is adjacent to w , w is adjacent to w_3 , $\alpha(w_2) = y_1^2$, $\alpha(w) = y$ and $\alpha(w_3) = y_1^3$. Since $v_0 - u_0 - u_1$ is the only pair of intersecting edges of X' mapped by φ onto $v_3 - v_0 - v_5$, we have the result that $\Phi_1^n(w_2) = v_0$, $\Phi_1^n(w) = u_0$ and $\Phi_1^n(w_3) = u_1$. By Proposition 3.4, there is a vertex b of Φ_n such that $w_3 \in \langle w_2, b \rangle$ and Φ_1^n maps $\langle w_2, b \rangle$ isomorphically onto $\langle v_0, v_2 \rangle$. Note that Φ_0^n maps the consecutive vertices of $\langle w_3, b \rangle$ onto the sequence v_5, v_0, v_2, v_0, v_1 . To match this pattern we must have that

$$(3.6.3) \quad \text{either } \beta(y_2^3) = v_0, \text{ or } \beta(y_2^1) = v_0 \text{ and } \beta(y_3^1) = v_1.$$

If $W_{1,2} = \emptyset$, then the proposition is true by 3.6.1. So we can assume that $W_{1,2} \neq \emptyset$. By Proposition 3.5, there are three vertices z_1 , z and z_2 of Φ_n such that z_1 is adjacent to z , z is adjacent to z_2 , $\alpha(z_1) = y_1^1$, $\alpha(z) = y$ and $\alpha(z_2) = y_1^2$. Since $v_0 - u_5 - u_6$ is the only pair of intersecting edges of X' mapped by φ onto $v_3 - v_0 - v_2$, we have the result that $\Phi_1^n(z_2) = v_0$, $\Phi_1^n(z) = u_5$ and $\Phi_1^n(z_1) = u_6$. By Proposition 3.4, there is a vertex c of Φ_n such that $z_1 \in \langle z_2, c \rangle$ and Φ_1^n maps $\langle z_2, c \rangle$ isomorphically onto $\langle v_0, v_3 \rangle$. Note that Φ_0^n maps the consecutive vertices of $\langle z_1, c \rangle$ onto the sequence v_2, v_0, v_5, v_6 . To match this pattern we must have that

$$(3.6.4) \quad \text{either } \beta(y_2^1) = v_0 \text{ and } \beta(y_3^1) = v_5, \text{ or } \beta(y_2^3) = v_6.$$

We will prove that

$$(3.6.5) \quad \text{either } \beta(y_2^1) = v_0 \text{ and } \beta(y_3^1) = v_1, \text{ or } \beta(y_2^3) = v_6.$$

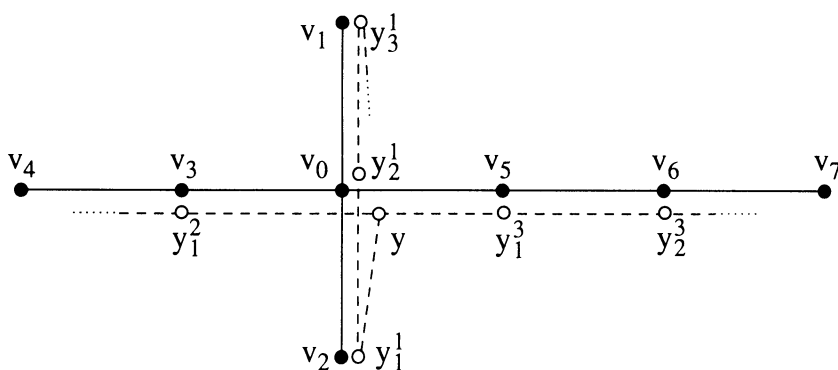


FIGURE 4

Again, by 3.6.1, we can assume that $W_{1,3} \neq \emptyset$. By Proposition 3.5, there are three vertices x_1 , x and x_3 of Φ_n such that x_1 is adjacent to x , x is adjacent to x_3 , $\alpha(x_1) = y_1^1$, $\alpha(x) = y$ and $\alpha(x_3) = y_1^3$. Since $u_6 - u_7 - u_8$ and $u_3 - u_2 - u_1$ are the only two pairs of intersecting edges of X' mapped by φ onto $v_2 - v_0 - v_5$, we have the result that either $\Phi_1^n(x_1) = u_6$, $\Phi_1^n(x) = u_7$ and $\Phi_1^n(x_3) = u_8$, or $\Phi_1^n(x_1) = u_3$, $\Phi_1^n(x) = u_2$ and $\Phi_1^n(x_3) = u_1$. Suppose that $\Phi_1^n(x_3) = u_8$. Then by Proposition 3.4, there is a vertex s of Φ_n such that s is adjacent to x_3 and $\Phi_1^n(s) = v_3$. This forces $\beta(y_2^3) = v_6$ and (3.6.5) is true. So, we can assume that $\Phi_1^n(x_1) = u_3$, $\Phi_1^n(x) = u_2$ and $\Phi_1^n(x_3) = u_1$. By Proposition 3.4, there is a vertex q of Φ_n such that $x_1 \in \langle x_3, q \rangle$ and Φ_1^n maps $\langle x_1, q \rangle$ isomorphically onto $\langle u_3, v_2 \rangle$. Note that Φ_0^n maps the consecutive vertices of $\langle x_1, q \rangle$ onto the sequence v_2, v_0, v_1 . To match this pattern we must have that $\beta(y_2^1) = v_0$ and $\beta(y_3^1) = v_1$. So (3.6.5) is true.

Combining (3.6.3), (3.6.4) and (3.6.5) we get the result that

$$(3.6.6) \quad \beta(y_2^1) = v_0, \quad \beta(y_3^1) = v_1 \text{ and } \beta(y_2^3) = v_6.$$

Figure 4 shows the map β on the “central” part of Y . As before, the dotted line graph represents the domain, Y , while the solid black represents the range, X , and each vertex of the domain is mapped onto the nearest vertex of the range.

Define $\mathcal{Y}'(Y')$ to be $\mathcal{Y}(Y)$ with y replaced by four points g_0, g_2, g_5 and g_7 , with y_1^1 replaced by two points g_3 and g_6 , and with y_1^3 replaced by two points g_1 and g_8 . Let $\mathcal{E}(Y')$ consists of $\langle y_2^1, g_3 \rangle$, $\langle y_2^3, g_8 \rangle$, $\langle y_1^2, g_0 \rangle$, $\langle y_1^2, g_5 \rangle$, $\langle g_0, g_1 \rangle$, $\langle g_1, g_2 \rangle$, $\langle g_2, g_3 \rangle$, $\langle g_5, g_6 \rangle$, $\langle g_6, g_7 \rangle$, $\langle g_7, g_8 \rangle$ and all edges of Y not containing any of the vertices y , y_1^2 and y_1^3 . Observe that Y' is a triod and y_1^2 is its vertex of order 3. (See Figure 5.) Define $\beta'(g_0) = \beta'(g_2) = \beta'(g_5) = \beta'(g_7) = v_0$, $\beta'(g_3) = \beta'(g_6) = v_2$, $\beta'(g_1) = \beta'(g_8) = v_5$ and $\beta'(w) = \beta(w)$ for $w \in \mathcal{Y}'(Y') \cap \mathcal{Y}(Y)$. Figure 5 shows the map β' on the “central” part of Y' . As usual, the dotted line graph represents the domain, Y' , while the solid black represents the range, X , and each vertex of the domain is mapped onto the nearest vertex of the range. Compare this figure with Figures 3 and 4.

Let G denote the set $\{v \in \mathcal{Y}(\Phi_n) | \alpha(v) \text{ is either } y, \text{ or } y_1^1, \text{ or } y_1^3\}$. Let

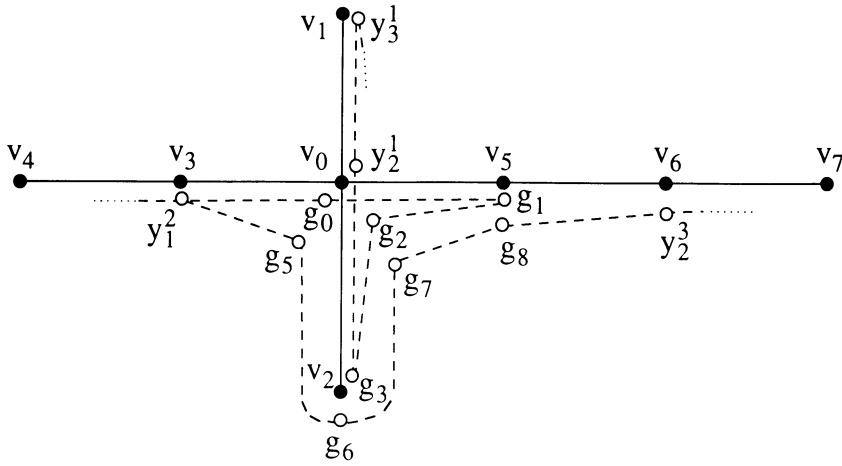


FIGURE 5

G_i denote the set $\{v \in G \mid \Phi_1^n(v) = u_i\}$ where $i = 0, 1, 2, 3, 5, 6, 7$ or 8 . Using 3.5.6 and Proposition 3.4 one can prove that $G = G_0 \cup G_1 \cup G_2 \cup G_3 \cup G_5 \cup G_6 \cup G_7 \cup G_8$. Define $\alpha'(v) = \alpha(v)$ for $v \in \mathcal{V}(\Phi_n) \setminus G$ and $\alpha'(v) = g_i$ for $v \in G_i$, $i = 0, 1, 2, 3, 5, 6, 7, 8$. Clearly, α' is a simplicial map and $\beta' \circ \alpha' = \Phi_0^n$. $\square$

4. THE OPERATION d

In this section we will recall combinatorial methods introduced in [8] and apply them to the map φ .

4.1 Definition. For a graph G_0 , Let $D(G_0)$ denote the graph such that

- (i) the set of vertices of $D(G_0)$ consists of edges of G_0 and
- (ii) two vertices of $D(G_0)$ are adjacent if and only if they intersect (as edges of G_0).

Let $\psi: G_1 \rightarrow G_0$ be a simplicial map between graphs. For every (closed) edge $e \in \mathcal{E}(G_0)$, let $\mathcal{K}(e)$ denote the set of components of $\psi^{-1}(e)$ which are mapped by ψ onto e . Denote by $\mathcal{K}(\psi)$ the union of all $\mathcal{K}(e)$. Let $D(\psi, G_1)$ be the graph such that

- (i) the vertices of $D(\psi, G_1)$ are elements of $\mathcal{K}(\psi)$, and
- (ii) two vertices of $D(\psi, G_1)$ are adjacent if and only if they intersect (as subgraphs of G_1).

Let $d[\psi]: D(\psi, G_1) \rightarrow D(G_0)$ be the map defined by the formula $d[\psi](v) = \psi(v)$ for every vertex v of $D(\psi, G_1)$.

Every vertex $v \in \mathcal{V}(D(\psi, G_1))$ is also a subgraph of G_1 . To avoid confusion we will denote this subgraph by v^* .

Let σ be simplicial maps of a graph G_2 into G_1 . Let $d[\psi, \sigma]: D(\psi \circ \sigma, G_2) \rightarrow D(\psi, G_1)$ be the map such that for every vertex v of $D(\psi \circ \sigma, G_2)$, $d[\psi, \sigma](v)$ is the vertex of $D(\psi, G_1)$ containing $\sigma(v^*)$.

4.2 Proposition. Suppose that Y is a triod which is the union of three arcs A_1 , A_2 and A_3 meeting at a common endpoint y . Let $y, y_1^i, y_2^i, \dots, y_{k(i)}^i$ denote the sequence of consecutive vertices of A_i . Suppose ψ is a simplicial map of Y into a graph G . Let p be the least integer such that $\psi(y_p^1) \neq \psi(y)$, and let q

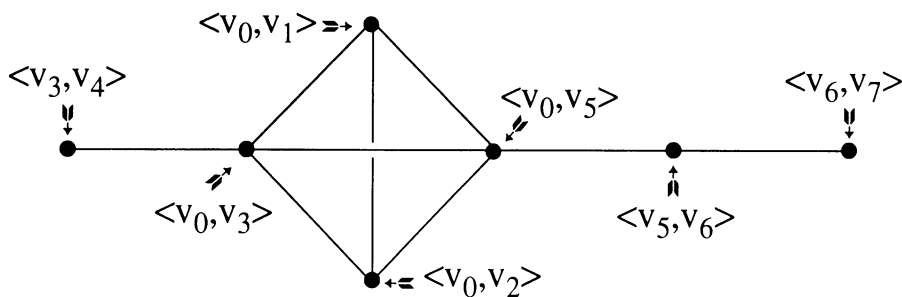


FIGURE 6

be the least integer such that $\psi(y_q^2) \neq \psi(y)$. If $\psi(y_p^1) = \psi(y_q^2)$, then $D(\psi, Y)$ is a triod (possibly degenerate).

Proof. Clearly, $\psi(\langle y, y_p^1 \rangle) = \psi(\langle y, y_q^2 \rangle)$ is an edge of G . Denote this edge by e . Let t be the vertex of $D(\psi, Y)$ representing the component of $\psi^{-1}(e)$ containing $\langle y, y_p^1 \rangle \cup \langle y, y_q^2 \rangle$. Observe that for any vertex $z \neq t$ of $D(\psi, Y)$, z^* is contained in one of the arcs A_1 , A_2 and A_3 . Let $\mathcal{Z}_1 = \{z \in D(\psi, Y) | z \neq t \text{ and } z^* \subset A_i\}$. Let z be an arbitrary point of $\mathcal{Z}_i$ and let j be an index such that $\langle y_j^i, y_{j+1}^i \rangle \subset z^*$ and $\psi(\langle y_j^i, y_{j+1}^i \rangle)$ is a nondegenerate edge of G . Observe that, if s is an element of $D(\psi, Y)$ different than z , then either $s^* \subset \langle y_{j+1}^i, y_{k(i)}^i \rangle$ or $s^* \subset Y \setminus \langle y_{j+1}^i, y_{k(i)}^i \rangle$. It follows that $\mathcal{Z}_i$ can be arranged into a sequence $z_1^i, z_2^i, \dots, z_{m(i)}^i$ such that $(z_1^i)^* \cap t^* \neq \emptyset$ and $(z_j^i)^* \cap (z_n^i)^* \neq \emptyset$ if and only if $|j - n| \leq 1$. Observe that $\langle t, z_{m(1)}^1 \rangle$, $\langle t, z_{m(2)}^2 \rangle$ and $\langle t, z_{m(3)}^3 \rangle$ are three arcs intersecting at t . Clearly, $D(\psi, Y)$ is the union of these arcs. $\square$

The following proposition follows immediately from 4.2.

4.3 Proposition. Suppose that Y is a triod with its point of order 3 denoted by y . Suppose β is a simplicial map of Y into X . If $\beta(y) \neq v_0$, then $D(\beta, Y)$ is a triod (possibly degenerate). $\square$

Figure 6 shows $D(X)$ with its vertices labeled by the corresponding to them edges of X . Figure 7 indicates $d[\varphi]: D(\varphi, X') \rightarrow D(X)$. The dotted line graph represents the domain, $D(\varphi, X')$, while the solid black is the range, $D(X)$, and each vertex of the domain is mapped onto the nearest vertex of the range. The vertices of $D(\varphi, X')$ are labeled $t_0, t_1, \dots, t_{12}$ as shown in Figure 7. Table 2 shows the subgraphs of X' corresponding to the vertices of $D(\varphi, X')$.

Let S be a function assigning to every vertex of X a set of edges of X defined in the following way: $S(v_0) = \{\langle v_0, v_2 \rangle, \langle v_0, v_3 \rangle, \langle v_0, v_5 \rangle\}$, $S(v_1) = \{\langle v_0, v_1 \rangle\}$, $S(v_2) = \{\langle v_0, v_2 \rangle\}$, $S(v_3) = \{\langle v_0, v_3 \rangle\}$, $S(v_4) = \{\langle v_3, v_4 \rangle\}$, $S(v_5) = \{\langle v_0, v_5 \rangle\}$, $S(v_6) = \{\langle v_5, v_6 \rangle\}$ and $S(v_7) = \{\langle v_6, v_7 \rangle\}$. Note that v_i belongs to each edge from $S(v_i)$ for $i = 0, \dots, 6$. So, S is an edge selection on X according to [8, Definition 5.5]. Observe that

(i) $\varphi((v_i e, X')) \in S(\varphi(v_i))$ for each $v_i \in \mathcal{V}(X)$ and each $e \in S(v_i)$, where (v_i, e, X') denote the edge of (e, X') containing v_i .

Observe also that

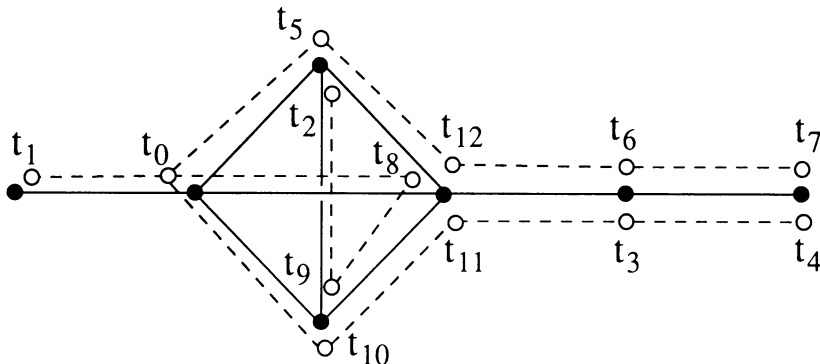


FIGURE 7

(ii) if e and e' are two different edges of X' intersecting at a common vertex q then at least one of the edges $\varphi(e)$ and $\varphi(e')$ belongs to $S(\varphi(q))$.

The above two conditions mean exactly that

4.4 Proposition. φ preserves (S, S) (in the sense of [8, Definition 5.7]). $\square$

Observe that

(iii) $(v_i, e, X') \subset t_i^*$ for each $i = 0, 1, \dots, 7$ and each $e \in S(v_i)$ and

(iv) $t_8^* = \langle u_0, u_1 \rangle \cup \langle u_1, u_2 \rangle \subset \langle v_0, v_2 \rangle$, $t_9^* = \langle u_2, u_3 \rangle \cup \langle u_3, u_4 \rangle \subset \langle v_0, v_2 \rangle$, $t_{10}^* = \langle u_5, u_6 \rangle \cup \langle u_6, u_7 \rangle \subset \langle v_0, v_3 \rangle$, $t_{11}^* = \langle u_7, u_8 \rangle \subset \langle v_0, v_3 \rangle$ and $t_{12}^* = \langle u_{10}, u_{11} \rangle \subset \langle v_5, v_6 \rangle$.

Let X'' be a subdivision of X with five new vertices: v_8 added between v_0 and v_2 , v_9 added between v_8 and v_2 , v_{10} added between v_0 and v_3 , and v_{12} added between v_{10} and v_3 , and v_{12} added between v_5 and v_6 . Let $\lambda: X'' \rightarrow D(\varphi, X')$ be defined by the formula $\lambda(v_i) = t_i$ for $i = 0, 1, \dots, 12$. Observe that λ is an isomorphism. Conditions (iii) and (iv) mean exactly that

TABLE 2

t_i	$d[\varphi](t_i)$	t_i^*
t_0	$\langle v_0, v_3 \rangle$	$\langle v_0, u_0 \rangle \cup \langle v_0, u_5 \rangle \cup \langle v_0, u_9 \rangle$
t_1	$\langle v_3, v_4 \rangle$	$\langle v_0, v_1 \rangle$
t_2	$\langle v_0, v_1 \rangle$	$\langle u_4, v_2 \rangle$
t_3	$\langle v_5, v_6 \rangle$	$\langle u_8, v_3 \rangle$
t_4	$\langle v_6, v_7 \rangle$	$\langle v_3, v_4 \rangle$
t_5	$\langle v_0, v_1 \rangle$	$\langle u_9, v_5 \rangle \cup \langle v_5, u_{10} \rangle$
t_6	$\langle v_5, v_6 \rangle$	$\langle u_{11}, v_6 \rangle$
t_7	$\langle v_6, v_7 \rangle$	$\langle v_6, v_7 \rangle$
t_8	$\langle v_0, v_5 \rangle$	$\langle u_0, u_1 \rangle \cup \langle u_1, u_2 \rangle$
t_9	$\langle v_0, v_2 \rangle$	$\langle u_2, u_3 \rangle \cup \langle u_3, u_4 \rangle$
t_{10}	$\langle v_0, v_2 \rangle$	$\langle u_5, u_6 \rangle \cup \langle u_6, u_7 \rangle$
t_{11}	$\langle v_0, v_5 \rangle$	$\langle u_7, u_8 \rangle$
t_{12}	$\langle v_0, v_5 \rangle$	$\langle u_{10}, u_{11} \rangle$

4.5 Proposition. φ is consistent on S and λ is a consistency isomorphism (see [8, Definition 5.7]). $\square$

The following proposition can be readily verified by Figure 7.

4.6 Proposition. $D[\varphi]$ is light and for each $e \in \mathcal{E}(D(X))$, each component of $(D[\varphi])^{-1}(e)$ is either a vertex or an edge of $D(\varphi, X')$. $\square$

5. K IS NOT SIMPLE-4-OD-LIKE

5.1 Proposition. Suppose Φ_0^n can be factored through a triod. Then the map $d[\varphi, \Phi_1^n] : D(\Phi_0^n, \Phi_n) \rightarrow D(\varphi, X')$ can also be factored through a triod.

Proof. Let Y be a triod with its point of order 3 denoted by y . Let $\alpha : \Phi_n \rightarrow Y$ and $\beta : Y \rightarrow X$ be simplicial maps such that $\alpha \circ \beta = \Phi_0^n$. By Proposition 3.5, we can assume that $\beta(y) \neq v_0$. It follows from Proposition 4.3, $D(\beta, Y)$ is a triod.

Observe that $d[\beta, \alpha] : D(\Phi_0^n, \Phi_n) \rightarrow D(\beta, Y)$, $D[\beta] : D(\beta, Y) \rightarrow D(X)$, $d[\varphi, \Phi_1^n] : D(\Phi_0^n, \Phi_n) \rightarrow D(\varphi, X')$ and $d[\varphi] : D(\varphi, X') \rightarrow D(X)$ are light simplicial maps. By Proposition 4.6, it follows from [8, Theorem 4.3] that there is $\beta' : D(\beta, Y) \rightarrow D(\varphi, X')$ such that $\beta' \circ d[\beta, \alpha] = d[\varphi, \Phi_1^n]$. $\square$

5.2 Proposition. The map Φ_0^n cannot be factored through a triod.

Proof. Clearly, the proposition is true if $n = 0$. Now, suppose that the proposition is true for $n - 1$. Let Γ denote the sequence $D[\varphi_1] \circ \lambda, \varphi_2, \varphi_3, \dots, \varphi_n$, where $\varphi_i = \varphi$, and let $\{\Gamma_j, \Gamma_j^i\}_{j=0}^n$ we denote the system generated by Γ . It follows from Propositions 4.4, 4.5 and [8, Theorem 5.11] that the system $\{D(\Phi_0^j, \Phi_j), d[\Phi_0^j, \Phi_j^i]\}_{j=0}^n$ is isomorphic to $\{\Gamma_j, \Gamma_j^i\}_{j=0}^n$.

Suppose Φ_0^n can be factored through a triod. Then, by Proposition 5.1, $d[\Phi_0^1, \Phi_1^n]$ and consequently Γ_1^n can be factored through a triod. Since the system $\{\Gamma_j, \Gamma_j^i\}_{j=1}^n$ is generated by subdivisions of $\varphi_2, \dots, \varphi_n$, according to the inductive assumption and Proposition 2.3, Γ_1^n cannot be factored through a triod. This contradiction proves the proposition. $\square$

5.3 Theorem. K is a simple-4-od-like but not simple-triod-like continuum and each proper nondegenerate subcontinuum of K is an arc.

Proof. K is simple-4-od-like as the inverse limit of the system $\{\Phi_j, \Phi_j^i\}$ of subdivisions of X . By Proposition 3.3, each proper nondegenerate subcontinuum of K is an arc. Suppose that K is triod-like. Then by Proposition 2.1, Φ_0^n can be factored through a triod for some n , contrary to Proposition 5.2. $\square$

REFERENCES

1. H. Cook, W. T. Ingram and A. Lelek, *Eleven annotated problems about continua*, Open Problems in Topology, North-Holland, Amsterdam, 1990, pp. 295–302.
2. J. F. Davies and W. T. Ingram, *An atriodic tree-like continuum with positive span which admits a monotone mapping to a chainable continuum*, Fund. Math. **131** (1988), 13–24.
3. W. T. Ingram, *Decomposable circle-like continua*, Fund. Math. **58** (1968), 193–198.
4. —, *An atriodic tree-like continuum with positive span*, Fund. Math. **77** (1972), 99–107.
5. —, *An uncountable collection of mutually exclusive planar atriodic tree-like continua with positive span*, Fund. Math. **85** (1974), 73–78.

6. A. Lelek, *Disjoint mappings and the span of spaces*, Fund. Math. **55** (1964), 199–214.
7. W. Lewis, *Continuum theory problems*, Topology Proc. **8** (1983), 361–394.
8. P. Minc, *On simplicial maps and chainable continua* (preprint).

FAT MATHEMATICS, AUBURN UNIVERSITY, AUBURN UNIVERSITY, ALABAMA 36849-5310

E-mail address: pminc@ducvax.auburn.edu